1. Let $A = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & -3 & 1 & 4 \end{pmatrix}$

   (a) Find an LU decomposition for A.
   (b) Find the determinants for L, U, and A.
   (c) Write L as a product of elimination matrices.
   (d) Write L^{-1} as a product of elimination matrices, or explain why L is not invertible.
   (e) Is A invertible? If so find the inverse. If not, explain how you know.
   (f) For a vector $\mathbf{b} \in \mathbb{R}^4$, how many solutions could $A\mathbf{x} = \mathbf{b}$ have?
   (g) Find an LU decomposition for A^T.

Solution:

(a) We do elimination on $A = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & -3 & 1 & 4 \end{pmatrix}$ to transform it to an upper triangular form.

First, subtract Row 1 from Row 4 by multiplying on the left by $E_{1,4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$ to get $\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -3 & 1 & 0 \end{pmatrix}$.

Then add 3(Row 2) to Row 4 by multiplying on the left by $E_{2,4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix}$ to get $\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

Then subtract (1/2)(Row 3) from Row 4 by multiplying on the left by $E_{3,4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/2 & 1 \end{pmatrix}$ to get the upper triangular matrix $U = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Summarizing this procedure, we obtain the equation $E_{3,4}E_{2,4}E_{1,4}A = U$, i.e.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & -3 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

To obtain the LU decomposition, we multiply by the inverses of the elimination matrices on the left in the appropriate order to obtain $A = E_{1,4}^{-1}E_{2,4}^{-1}E_{3,4}^{-1}U$, where

$$E_{1,4}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad E_{2,4}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{pmatrix}, \quad E_{3,4}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 1 \end{pmatrix}.$$

Thus we have the LU decomposition

$$\begin{aligned} A &= \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & -3 & 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -3 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

(b) Since the determinant of a triangular matrix is the product of the entries on the diagonal, $\det L = 1$ and $\det U = 0$. Since the determinant is multiplicative, $\det A = \det L \det U = 1 \cdot 0 = 0$.

(c)

$$L = E_{1,4}^{-1}E_{2,4}^{-1}E_{3,4}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 1 \end{pmatrix}.$$

(d)

$$L^{-1} = E_{3,4}E_{2,4}E_{1,4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

(e) Since $\det A = 0$, A is not invertible.

(f) Since A is not invertible, it has a nontrivial nullspace. Thus if $A\mathbf{x} = \mathbf{b}$ has a solution, there will be an infinite number of solutions.

However since A is not invertible it does not have full row rank, so $A\mathbf{x} = \mathbf{b}$ may not have any solutions (A has rank 3, so the column space is a 3-dimensional subspace of $\mathbb{R}^4$).

(g)

$$A^T = (LU)^T = U^T L^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

2. Let $A = \begin{pmatrix} 2 & 2 & -4 \\ 0 & 1 & -3 \\ -2 & -2 & 2 \\ 2 & 1 & -5 \\ 0 & -1 & -1 \end{pmatrix}$.

(a) Find the reduced row echelon form R for A.
(b) Find the rank of A and the dimensions of the four fundamental subspaces.
(c) Find a basis for the column space of A.
(d) Find a vector $\mathbf{b} \in \mathbb{R}^5$ such that $R\mathbf{x} = \mathbf{b}$ has no solutions, or explain why this is not possible.
(e) Find a vector $\mathbf{b} \in \mathbb{R}^5$ such that $R\mathbf{x} = \mathbf{b}$ has one solution, or explain why this is not possible.
(f) Find a vector $\mathbf{b} \in \mathbb{R}^5$ such that $R\mathbf{x} = \mathbf{b}$ has an infinite number of solutions, or explain why this is not possible.
(g) Is A^TA positive definite, positive semi-definite, or neither?
(h) Is AA^T positive definite, positive semi-definite, or neither?
(i) Is $C(R) = C(A)$?
(j) Is $N(R) = N(A)$?

Solution:

(a) Beginning with $A = \begin{pmatrix} 2 & 2 & -4 \\ 0 & 1 & -3 \\ -2 & -2 & 2 \\ 2 & 1 & -5 \\ 0 & -1 & -1 \end{pmatrix}$, we do elimination to transform it to RREF.

Adding Row 1 to Row 3, and then subtracting Row 1 from Row 4 we obtain $\begin{pmatrix} 2 & 2 & -4 \\ 0 & 1 & -3 \\ 0 & 0 & -2 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}$

Subtracting twice Row 2 from Row 1, and then adding Row 2 to Rows 4 and 5 we obtain $\begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & -2 \\ 0 & 0 & -4 \\ 0 & 0 & -4 \end{pmatrix}$

Multiplying Row 1 by $1/2$ and multiplying Row 3 by $-1/2$, we obtain $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & -4 \\ 0 & 0 & -4 \end{pmatrix}$

Subtracting Row 3 from Row 1, adding 3(Row 3) to Row 2, and subtracting 4(Row 3) from Rows 4 and 5, we obtain the RREF $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(b) The rank of A is $r = 3$. By the rank-nullity theorem,

- $\dim C(A) = r = 3$
- $\dim C(A^T) = r = 3$
- $\dim N(A) = n - r = 3 - 3 = 0$ (trivial nullspace)
- $\dim N(A^T) = m - r = 5 - 3 = 2$

(c) Since A has full column rank, the columns of A form a basis for the column space of A. I.e. a basis for $C(A)$ is $\{(2, 0, -2, 2, 0), (2, 1, -2, 1, -1), (-4, -3, 2, -5, -1)\}$.

(d) If $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$, then $R\mathbf{x} = \mathbf{b}$ has no solutions if and only if x_4 or x_5 is nonzero.

(e) If $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$, then $R\mathbf{x} = \mathbf{b}$ has a solution if and only if $x_4 = x_5 = 0$. Since R has full column rank (trivial nullspace), if $R\mathbf{x} = \mathbf{b}$ has a solution, that solution is unique.

(f) Since R has full column rank (trivial nullspace), if $R\mathbf{x} = \mathbf{b}$ has a solution, there is only one solution. so there can never be an infinite number of solutions.

(g) Since the columns of A are independent, A^TA is positive definite.

(h) Since the columns of A^T (rows of A) are not independent, AA^T is positive semi-definite.

(i) $C(R) \neq C(A)$. For example, $(2, 0, -2, 2, 0) \in C(A)$, but $(2, 0, -2, 2, 0) \notin C(R)$. (Both column spaces are 3-dimensional subspaces of $\mathbb{R}^5$, but they are different subspaces).

(j) Since row operations leave $\mathbf{0}$ unchanged, $A\mathbf{x} = \mathbf{0}$ if and only if $R\mathbf{x} = \mathbf{0}$. So $N(R) = N(A)$.

3. (a) Define what it means for a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ to be linearly independent.

   (b) Define the span of a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

   (c) Define what it means for a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ to be a basis of a vector space V.

   (d) Use the definitions of linear independence, span, and basis to prove the following: Given a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for a vector space V, every vector in V can be written as a linear combination of the basis vectors in exactly one way.

Solution:

(a) A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent if the only linear combination of these vectors equal to $\mathbf{0}$ is the trivial one. I.e.

$$c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n = \mathbf{0} \iff c_1 = \cdots = c_n = 0.$$

(b) The span of a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is the set of linear combinations of these vectors. I.e.

$$\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = \{c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n : c_1, \ldots, c_n \in \mathbb{R}\}.$$

(c) A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of a vector space V if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent and $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = V$.

(d) Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, and let $\mathbf{v}$ be an arbitrary vector in V.

Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis, $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = V$, so by the definition of span, there are scalars $a_1, \ldots, a_n$ such that

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{v}$$

We now show that this representation is unique. Suppose there are scalars $b_1, \ldots, b_n$ such that

$$b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n = \mathbf{v}.$$

Then subtracting the two equations we obtain

$$(a_1 - b_1)\mathbf{v}_1 + \cdots + (a_n - b_n)\mathbf{v}_n = \mathbf{v} - \mathbf{v} = \mathbf{0}.$$

Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis, this set is linearly independent, so each coefficient $(a_i - b_i)$ in the linear combination equal to $\mathbf{0}$ must be equal to 0. Thus $a_i = b_i$ for all i, and there is only one linear combination of the basis vectors equal to $\mathbf{v}$

4. Let $\mathbf{v}_1 = (1, 3, 1)$, $\mathbf{v}_2 = (-2, 1, -1)$, and $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Let $\mathbf{w} = (-4, 1, 0)$.

   (a) Verify that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal.

   (b) Find an orthonormal basis for V.

   (c) Find a matrix P such that for all $\mathbf{b} \in \mathbb{R}^3$, $P\mathbf{b}$ is the projection of $\mathbf{b}$ onto V. You may leave your answer P as a product of matrices.

   (d) Find the projection of $\mathbf{w}$ onto $\mathbf{v}_1$.

   (e) Find the projection of $\mathbf{w}$ onto $\mathbf{v}_2$.

   (f) Find the projection of $\mathbf{w}$ onto V.

   (g) Is the angle made by $\mathbf{v}_1$ and $\mathbf{w}$ acute, obtuse, or right?

   (h) Is the angle made by $\mathbf{v}_2$ and $\mathbf{w}$ acute, obtuse, or right?

   (i) Find a vector $\mathbf{x}$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w} - \mathbf{x}\}$ is an orthogonal basis for $\mathbb{R}^3$.

Solution:

(a) We check that the dot product is zero:

$$\mathbf{v}_1^T \mathbf{v}_2 = (1)(-2) + (3)(1) + (1)(-1) = -2 + 3 - 1 = 0.$$

(b) An orthonormal basis for V is given by $\left\{ \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} \right\} = \left\{ \frac{1}{\sqrt{11}}(1, 3, 1), \frac{1}{\sqrt{6}}(-2, 1, -1) \right\}$.

(c) We give two equivalent answers. In both, we use the fact that $P = A(A^TA)^{-1}A^T$, where A is a matrix whose columns form a basis for V.
Answer 1: Let A be the matrix whose columns are the orthonormal vectors in Part (b). Then $A^TA = I_2$, so $(A^TA)^{-1} = I_2$, and

$$P = A(A^TA)^{-1}A^T = AA^T = \begin{pmatrix} 1/\sqrt{11} & -2/\sqrt{6} \\ 3/\sqrt{11} & 1/\sqrt{6} \\ 1/\sqrt{11} & -1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1/\sqrt{11} & 3/\sqrt{11} & 1/\sqrt{11} \\ -2/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \end{pmatrix}$$

Answer 2: Let A be the matrix whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$. Then $A^TA = \begin{pmatrix} 11 & 0 \\ 0 & 6 \end{pmatrix}$, and $(A^TA)^{-1} = \begin{pmatrix} 1/11 & 0 \\ 0 & 1/6 \end{pmatrix}$. Then

$$P = A(A^TA)^{-1}A^T = AA^T = \begin{pmatrix} 1 & -2 \\ 3 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1/11 & 0 \\ 0 & 1/6 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ -2 & 1 & -1 \end{pmatrix}.$$

(d)

$$\text{proj}_{\mathbf{v}_1} \mathbf{w} = \frac{\mathbf{w}^T \mathbf{v}_1}{\mathbf{v}_1^T \mathbf{v}_1} \mathbf{v}_1 = \frac{-1}{11}(1, 3, 1).$$

(e)

$$\text{proj}_{\mathbf{v}_2} \mathbf{w} = \frac{\mathbf{w}^T \mathbf{v}_2}{\mathbf{v}_2^T \mathbf{v}_2} \mathbf{v}_2 = \frac{9}{6}(-2, 1, -1).$$

(f) Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal the projection of $\mathbf{w}$ onto V is

$$\text{proj}_{\mathbf{v}_1}\mathbf{w} + \text{proj}_{\mathbf{v}_2}\mathbf{w} = \frac{-1}{11}(1,3,1) + \frac{9}{6}(-2,1,-1).$$

(g) Since the dot product $\mathbf{v}_1^T\mathbf{w} < 0$, the angle made by $\mathbf{v}_1$ and $\mathbf{w}$ is obtuse.

(h) Since the dot product $\mathbf{v}_2^T\mathbf{w} > 0$, the angle made by $\mathbf{v}_2$ and $\mathbf{w}$ is acute.

(i) The vector $\mathbf{w} - \text{proj}_{\mathbf{v}_1}\mathbf{w} - \text{proj}_{\mathbf{v}_2}\mathbf{w}$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v_2}$, and is nonzero. Thus, setting

$$\mathbf{x} = \text{proj}_{\mathbf{v}_1}\mathbf{w} + \text{proj}_{\mathbf{v}_2}\mathbf{w} = \frac{-1}{11}(1,3,1) + \frac{9}{6}(-2,1,-1),$$

the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w} - \mathbf{x}\}$ is an orthogonal basis for $\mathbb{R}^3$.

5. Consider the permutation matrix $P = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$.

(a) What is its determinant?

(b) What is its inverse?

(c) Is P a Markov matrix?

(d) Find an eigenvector $\mathbf{v}$ of P with eigenvalue 1. Is $\mathbf{v}$ an attracting steady state?

(e) Suppose $\mathbf{v}$ is a vector with $\|\mathbf{v}\| = 5$. Find $\|P\mathbf{v}\|$.

(f) Suppose U is an upper triangular matrix. Which of the entries on or above the diagonal of PU must be 0?

(g) Suppose U is an upper triangular matrix. Which of the entries on or above the diagonal of UP must be 0?

(h) Suppose $\mathbf{v} = (1, 2, 3, 4, 5)$. What is $P^{2021}\mathbf{v}$?

Solution:

(a) Since it requires an even number of row swaps to change P to I_5 (e.g. R1/R2, R2/R3, R3/R5, R4/R5), $\det P = 1$.

(b) Since P is an orthogonal matrix,

$$P^{-1} = P^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(c) Since all entries are nonnegative, and the entries in each column sum to 1, P is a Markov matrix.

(d) If$\mathbf{v} = (1, 1, 1, 1, 1)$, then $P\mathbf{v} = \mathbf{v}$, so $\mathbf{v}$ is an eigenvector of P with eigenvalue 1. This vector is not an attracting steady state: For any vector $\mathbf{x}$, $P\mathbf{x}$ has the same components as $\mathbf{x}$, just permuted, so in general the entries of $P^k\mathbf{x}$ will not converge to be all the same.

(e) Since $P^TP = I$,

$$\|P\mathbf{v}\| = \sqrt{(P\mathbf{v})^T(P\mathbf{v})} = \sqrt{\mathbf{v}^TP^TP\mathbf{v}} = \sqrt{\mathbf{v}^T\mathbf{v}} = \|\mathbf{v}\| = 5.$$

(f) PU is a matrix with the rows of U permuted. $(PU)_{11} = (PU)_{12} = (PU)_{13} = (PU)_{44} = 0$:

$$PU = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & 0 & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & 0 & x \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & x & x \\ x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & 0 & 0 & 0 & x \\ 0 & 0 & x & x & x \end{pmatrix}$$

(g) UP is a matrix with the columns of U permuted. $(UP)_{24} = (UP)_{34} = (UP)_{44} = (UP)_{55} = 0$:

$$UP = \begin{pmatrix} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & 0 & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & 0 & x \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} x & x & x & x & x \\ x & x & x & 0 & x \\ 0 & x & x & 0 & x \\ 0 & 0 & x & 0 & x \\ 0 & 0 & x & 0 & 0 \end{pmatrix}$$

(h) Since $P^5 = I$, $P^{2021}\mathbf{v} = P\mathbf{v} = (4, 1, 2, 5, 3)$.

6. Consider the matrix $A = \begin{pmatrix} 8 & 2 \\ 2 & 8 \end{pmatrix}$.

(a) Find the characteristic polynomial for A.

(b) Find the eigenvectors and eigenvalues for A.

(c) Find an orthogonal diagonalization for A.

(d) Is A positive definite, positive semi-definite, or neither?

(e) Is A invertible? If so, give a diagonalization for A^{-1}. Otherwise explain why not.

(f) Find a diagonalization for a matrix B such that $B^2 = A$, or explain why no such matrix exists.

(g) Give a diagonalization for A^{16}.

(h) What is the determinant of A^{16}?

Solution:

(a) The characteristic polynomial for A is

$$\det \begin{pmatrix} 8-\lambda & 2 \\ 2 & 8-\lambda \end{pmatrix} = (8-\lambda)^2 - 2^2 = \lambda^2 - 16\lambda + 60 = (\lambda - 10)(\lambda - 6).$$

(b) The eigenvalues for A are $\lambda_1 = 10$ and $\lambda_2 = 6$. The corresponding eigenvectors are $\mathbf{x}_1 = (1, 1)$ and $\mathbf{x}_2 = (-1, 1)$.

(c) For an orthogonal diagonalization $A = Q\Lambda Q^T$, we normalize the eigenvectors to give an orthonormal basis of eigenvectors:

$$\begin{pmatrix} 8 & 2 \\ 2 & 8 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

(d) Since all eigenvalues of A are positive, A is positive definite.

(e) Since $\det A > 0$, A is invertible, and the inverse has diagonalization

$$A^{-1} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/10 & 0 \\ 0 & 1/6 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

(Same eigenvectors, eigenvalues are reciprocal).

(f) A matrix B such that $B^2 = A$, is given by

$$B = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{6} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

(g)

$$A^{16} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 10^{16} & 0 \\ 0 & 6^{16} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

(h) The determinant of A^{16} is equal to $(\det A)^{16}$, which is 60^{16}.

7. Suppose the matrix A has the following singular vectors and singular values:

   Singular vectors:

   - $\mathbf{v}_1 = (1/2, 1/2, 0, 1/2, 1/2)$, $\mathbf{v}_2 = (1/\sqrt{2}, -1/\sqrt{2}, 0, 0, 0)$, $\mathbf{v}_3 = (0, 0, 0, -1/\sqrt{2}, 1/\sqrt{2})$, $\mathbf{v}_4 = (0, 0, 1, 0, 0)$, $\mathbf{v}_5 = (-1/2, -1/2, 0, 1/2, 1/2)$
   - $\mathbf{u}_1 = (3/5, 4/5)$, $\mathbf{u}_2 = (-4/5, 3/5)$

   Singular values: $\sigma_1 = 7$ and $\sigma_2 = 3$.

   (a) Write the full format singular value decomposition for A.
   (b) How many rows and columns does A have?
   (c) What is the rank r of A?
   (d) Write the rank r singular value decomposition for A.
   (e) Write A as a sum of r rank 1 matrices given by the singular value decomposition.
   (f) Find the rank 1 approximation for A given by the singular value decomposition.
   (g) Write the singular value decomposition of A^T.
   (h) Diagonalize AA^T.
   (i) Suppose $\mathbf{v} = 3\mathbf{v}_1 - 5\mathbf{v}_2 + 13\mathbf{v}_3 - 81\mathbf{v}_4 + 179\mathbf{v}_5$. Find $A\mathbf{v}$.
   (j) For $\mathbf{b} \in \mathbb{R}^2$, could $A\mathbf{x} = \mathbf{b}$ have 0, 1, or an infinite number of solutions? Explain how you know.

Solution:

(a)

$$A = U\Sigma V^T = \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix} \begin{pmatrix} 7 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & 0 & 1/2 & 1/2 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1 & 0 & 0 \\ -1/2 & -1/2 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

(b) A is 2×5.

(c) The rank of A is $r = 2$.

(d) The rank 2 format singular value decomposition for A is

$$A = U_2\Sigma_2 V_2^T = \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & 0 & 1/2 & 1/2 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 & 0 \end{pmatrix}.$$

(e)

$$\begin{aligned} A &= \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T \\ &= 7 \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & 0 & 1/2 & 1/2 \end{pmatrix} + 3 \begin{pmatrix} -4/5 \\ 3/5 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 21/10 & 21/10 & 0 & 21/10 & 21/10 \\ 28/10 & 28/10 & 0 & 28/10 & 28/10 \end{pmatrix} + \begin{pmatrix} -12/5\sqrt{2} & 12/5\sqrt{2} & 0 & 0 & 0 \\ 9/5\sqrt{2} & -9/5\sqrt{2} & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

(f) The rank 1 approximation for A given by the singular value decomposition is

$$A_1 = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T = 7 \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & 0 & 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 21/10 & 21/10 & 0 & 21/10 & 21/10 \\ 28/10 & 28/10 & 0 & 28/10 & 28/10 \end{pmatrix}.$$

(g) The singular value decomposition of A^T is

$$A^T = V \Sigma^T U^T = \begin{pmatrix} 1/2 & 1/\sqrt{2} & 0 & 0 & -1/2 \\ 1/2 & -1/\sqrt{2} & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 0 \\ 1/2 & 0 & -1/\sqrt{2} & 0 & 1/2 \\ 1/2 & 0 & 1/\sqrt{2} & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 0 & 3 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix}.$$

(h) Since $AA^T = (U\Sigma V^T)(V\Sigma^T U^T)$ and $V^T V = I$,

$$AA^T = U\Sigma\Sigma^T U^T = \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix} \begin{pmatrix} 49 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix}.$$

(i)

$$\begin{aligned} A\mathbf{v} &= A(3\mathbf{v}_1 - 5\mathbf{v}_2 + 13\mathbf{v}_3 - 81\mathbf{v}_4 + 179\mathbf{v}_5) \\ &= 3A\mathbf{v}_1 - 5A\mathbf{v}_2 + 13A\mathbf{v}_3 - 81A\mathbf{v}_4 + 179A\mathbf{v}_5 \\ &= 3(7\mathbf{u}_1) - 5(3\mathbf{u}_2) + 13(\mathbf{0}) - 81(\mathbf{0}) + 179(\mathbf{0}) \\ &= 21(3/5, 4/5) - 15(-4/5, 3/5) \\ &= (123/5, 36/5) \end{aligned}$$

(j) Since A has full row rank, and a 3-dimensional null space, for all $\mathbf{b} \in \mathbb{R}^2$, $A\mathbf{x} = \mathbf{b}$ has an infinite number of solutions.